Thin Film Fluid Equation Derivation For Flat Surfaces

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Abstract

The Thin Film Fluid Equation is derived from the Navier Stokes Equation assuming fluid motion on a flat substrate.

1 Derivation of the Thin Film Equation

The thin film equation will be derived. First, the boundary conditions that will be used in the derivation will be listed.

1.1 Boundary Conditions

1.1.1 Kinetic Equation

Let $\phi = h(x,t) - y$ so that the 0 level set of ϕ describes the thin film surface. We want this statement to be valid for all times, which happens iff $\frac{D\phi}{Dt} = 0$ Where $\frac{D}{Dt}$ represents the material derivative. Plugging in the value for ϕ yields so called "Kinetic Equation":

$$h_t + uh_x = v \tag{1}$$

1.1.2 Surface Tension Pressure

At the surface should be proportional to the negative of the curvature ($p \propto -\kappa$) of the fluid surface in order to drive the surface to minimize area. The fact that it is the Surface Curvature is stressed. In the simple setting of a flat 2D plane we formulate this condition as:

$$p(h) = p_0 - \gamma\kappa \tag{2}$$

where p_0 is a constant.

*This is for making an acknowledgement.

1.1.3 Free Surface

On the surface boundary, the fluid is free to flow:

$$u_y(h) = 0 \tag{3}$$

1.1.4 No Slip

A no-slip condition where the fluid is touching the substrate is also imposed.

$$u(0) = v(0) = 0 \tag{4}$$

1.1.5 Continuity Equation

Since we will be dealing with a Newtonian fluid, the divergence of the velocity field has to be zero. This is equivalent to saying that the volume of the fluid will be conserved.

$$\nabla \cdot \vec{u} = u_x + v_y = 0$$

Another form of this equation which is will be useful at a later step is

$$v(h) - v(0) = v(h) = \int_0^h -u_x dy$$

which can be combined with the kinetic equation to obtain

$$h_t + uh_x = \int_0^h -u_x dy \tag{5}$$

1.2 Non-dimensionalizing Navier-Stokes

Writing the Navier-Stokes equations for each dimension, for a fluid including viscosity forces results in the following pair of equations:

$$u_t + uu_x + vu_y = -\frac{1}{\rho}p_x + \frac{\mu}{\rho}(u_{xx} + u_{yy}) \tag{6}$$

$$v_t + uv_x + vv_y = -\frac{1}{\rho}p_y + \frac{\mu}{\rho}(v_{xx} + v_{yy}) + g \tag{7}$$

For non-dimesionalizing purposes we will use $x = L\tilde{x}$, $y = \epsilon L\tilde{y}$, $u = U\tilde{u}$, $v = \epsilon U\tilde{v}$, and $p = \frac{\mu U^2}{\epsilon^2 L}\tilde{p}$. Replacing all these values into 6 and 7 one obtains:

$$\frac{U^2}{L}(\tilde{u}_{\tilde{t}} + \tilde{u}\tilde{u}_{\tilde{x}} + \tilde{v}\tilde{u}_{\tilde{y}}) = \frac{\mu U}{\epsilon^2 \rho L^2}(-\tilde{p}_{\tilde{x}} + \epsilon^2 \tilde{u}_{\tilde{x}\tilde{x}} + \tilde{u}_{\tilde{y}\tilde{y}})$$

$$\frac{\epsilon U^2}{L}(\tilde{v}_{\tilde{t}} + \tilde{u}\tilde{v}_{\tilde{x}} + \tilde{v}\tilde{v}_{\tilde{y}}) = \frac{\mu U}{\epsilon^3 \rho L^2}(-\tilde{p}_{\tilde{y}} + \epsilon^4 \tilde{v}_{\tilde{x}\tilde{x}} + \epsilon^2 \tilde{v}_{\tilde{y}\tilde{y}}) + g$$

Assuming that the Reynolds number $Re = \frac{\rho LU}{\mu} << 1$ one can rewrite the previous equations as:

$$\epsilon^2 Re(\tilde{u}_{\tilde{t}} + \tilde{u}\tilde{u}_{\tilde{x}} + \tilde{v}\tilde{u}_{\tilde{y}}) = -\tilde{p}_{\tilde{x}} + \epsilon^2 \tilde{u}_{\tilde{x}\tilde{x}} + \tilde{u}_{\tilde{y}\tilde{y}}$$

$$\epsilon^4 Re(\tilde{v}_{\tilde{t}} + \tilde{u}\tilde{v}_{\tilde{x}} + \tilde{v}\tilde{v}_{\tilde{y}}) = -\tilde{p}_{\tilde{y}} + \epsilon^4 \tilde{v}_{\tilde{x}\tilde{x}} + \epsilon^2 \tilde{v}_{\tilde{y}\tilde{y}} + g$$

From the previous equations, one can argue that an approximation in the Thin Film scenario where $Re\epsilon^2 << 1$ and $\epsilon << 1$ of the N-S equations is the following so called "Thin Film Approximation":

$$p_x = \mu u_{yy} \tag{8}$$

$$p_y = \rho g \tag{9}$$

1.3 Thin Film Equations

In this section, 8 and 9 will be manipulated and all the boundary conditions will be used to come up with a single PDE that is equivalent.

Equation 9 can be integrated:

$$\int_y^h p_y dy = \int_y^h \rho g dy$$

$$p(h) - p(y) = \rho g h - \rho g y$$

Combining this with 2 gives:

$$p(y) = -\rho g h + \rho g y + p_0 - \gamma\kappa$$

This expression for the pressure can be plugged into 8 so that:

$$-\rho g h_x + \gamma\kappa_x = p_x = \mu u_{yy}$$

Notice that p_x does not depend on y, so it can easily be integrated. Integrating the last equation yields

$$p_x h - p_x y = \int_y^h p_x dy = \mu \int_y^h u_y y dy = \mu(u_y(h) - u_y(y)) = -\mu u_y(y)$$

where we used the Free Surface Flow boundary condition (3) to drop the last term. We can integrate this equation again, which gives

$$p_x h y - \frac{p_x y^2}{2} = \int_0^y p_x h - p_x y dy = -\mu \int_0^y u_y dy = -\mu(u(y) - u(0)) = -\mu u(y)$$

where we used the No-slip boundary condition 4. This equation is known as the "Quadratic velocity profile" equation

$$u(y) = \frac{1}{\mu}(\frac{p_x y^2}{2} - p_x h y) \tag{10}$$

At this point, the only pair of boundary conditions we haven't used are the Continuity and Kinetic equations. Previously, we had combined both of them to obatin Eq. 5. Plugging the Quadratic Velocity Profile into 5 and using the fact that:

$$u_x = \frac{1}{\mu}(\frac{p_{xx}y^2}{2} - p_{xx}hy - p_x h_x y)$$

gives rise to

$$h_t + \frac{h_x}{\mu}(\frac{p_x h^2}{2} - p_x h^2) = \int_0^h -u_x dy$$

$$= \int_0^h (\frac{1}{\mu}(p_{xx}hy + p_x h_x y - \frac{p_{xx}y^2}{2}))dy$$

$$= \frac{p_{xx}h^3}{2\mu} + \frac{p_x h_x h^2}{2\mu} - \frac{p_{xx}h^3}{6\mu}$$

which when you further simplify

$$h_t = \frac{1}{\mu}(\frac{1}{3}p_{xx}h^3 + p_x h_x h^2) = \frac{1}{3\mu}\frac{\partial}{\partial x}(p_x h^3) = \frac{1}{3\mu}\frac{\partial}{\partial x}(\rho g h_x h^3 - \gamma \kappa_x h^3)$$

so that in the end you are left with the Thin Film Fluid Equation

$$h_t = \frac{1}{3\mu}\frac{\partial}{\partial x}(\rho g h_x h^3 - \gamma h_{xxx} h^3) \tag{11}$$